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HARDY-SOBOLEV TYPE INEQUALITIES ON HOMOGENEOUS GROUPS AND APPLICATIONS

ABSTRACT

of the PhD thesis for the degree of
doctor of Philosophy (PhD) in the specialty
«6D060100-Mathematics»

The relevance of the research topic. The subject of Hardy inequalities has now been a fascinating subject of continuous research by numerous mathematicians for exactly one century, 1918-2018. The original inequality was published by G. H. Hardy in “Notes on some points in the integral calculus (51)”, Messenger of Mathematics, 48 (1918), P. 107-112.

Over the last 100 years, the subject of Hardy inequalities and related analysis has been a topic of intensive research: currently MathSciNet lists more than 800 papers containing words ‘Hardy inequality’ in the title, and almost 3500 papers containing words ‘Hardy inequality’ in the abstract or in the review. The Hardy inequalities have been already presented in many monographs and reviews; here we can mention excellent presentations by Opic and Kufner in 1990, Davies in 1999, Edmunds and Evans in 2004, part of Mazya's books, Ghoussoub and Moradifam in 2013, and Balinsky, Evans and Lewis in 2015, as well as books on different areas related to Hardy spaces: Hardy inequalities on time scales, Hardy inequalities with general kernels, weighted Hardy inequalities, Hardy inequalities and sequence spaces.

However, all of these presentations are largely confined to the Euclidean part of the available wealth of information on this subject.

At the same time, there is another layer of intensive research over the years related to Hardy type functional inequalities in subelliptic settings motivated by their applications to problems involving subelliptic differential equations. This is complemented by the more general anisotropic versions of the theory. In this direction, the subelliptic ideas of the analysis on the Heisenberg group, significantly advanced by Folland and Stein in 1974, were subsequently consistently developed by Folland in 1975 leading to the foundations for analysis on stratified groups (or homogeneous Carnot groups). Furthermore, in their fundamental book titled “Hardy spaces on homogeneous groups”, Folland and Stein laid down foundations for the ‘anisotropic’ analysis on general homogeneous

groups, i.e. Lie groups equipped with a compatible family of dilations. Such groups are necessarily nilpotent, and the realm of homogeneous groups almost exhausts the whole class of nilpotent Lie groups including the classes of stratified, and more generally, graded groups. Among many, one of the motivations behind doing analysis on homogeneous groups is the “distillation of ideas and results of harmonic analysis depending only on the group and dilation structures”. The place where Hardy type inequalities and homogeneous groups meet is a beautiful area of mathematics which was not consistently treated in the PhD thesis. The main purpose of the PhD thesis extends and deepens the understanding of Hardy type inequalities and closely related topics from the point of view of Folland and Stein's homogeneous groups. While we will construct the general theory of Hardy, Rellich, Caffarelli-Kohn-Nirenberg, Hardy-Sobolev and other inequalities in the setting of stratified Lie groups. In this setting, the theory of subelliptic functional inequalities becomes intricately intertwined with the properties of sub-Laplacians and more general subelliptic partial differential equations.

These topics constitute the core of this PhD thesis with the results complemented with additional closely related topics such as uncertainty principles, theory of linear and nonlinear subelliptic differential equations as well as theory of (anisotropic) functional spaces.

The aim of the PhD thesis is to study the subelliptic functional inequalities on homogenous groups such as Hardy, Hardy-Sobolev, Caffarelli-Kohn-Nirenberg, and Rellich inequalities.

Objectives of the study:

- establishing geometric Hardy and Hardy-Sobolev inequalities on the half-spaces of the stratified groups;
- obtaining horizontal functional inequalities such as Hardy, Rellich, Caffarelli-Kohn-Nirenberg, Hardy-Rellich inequalities;
- obtaining weighted Hardy and Rellich inequalities with boundary terms for the sub-Laplacian fundamental solution;
- obtaining weighted Hardy and Rellich inequalities for general vector fields.

Object of the PhD thesis is the proof of the main functional inequalities for subelliptic differential equations, namely, subelliptic extension of Hardy, Caffarelli-Kohn-Nirenberg, Rellich, Hardy-Sobolev inequalities, and investigation of ‘subelliptic’ functional spaces and embedding theorems.

The methods of scientific research. For research on this PhD thesis, the classical methods of calculus, the theory of Lie groups, theory of PDE, along with new ideas of mathematical sciences are used.

Scientific novelty of the work. The following new results were obtained in the PhD thesis:

- *Geometric Hardy and Hardy-Sobolev inequalities on the stratified groups.* In this direction, we obtained the geometric Hardy and Hardy-Sobolev inequalities on the half-spaces. We presented L^2 and L^p versions of the (subelliptic) geometric Hardy inequalities in half-spaces and convex domains on general stratified groups. As a consequence, we have derived the Hardy-Sobolev inequality in the half-space on the Heisenberg group. Moreover, the geometric Hardy inequality on the starshaped sets is established.

- *Horizontal inequalities on the stratified groups.* In the second direction, we study the following horizontal version of Hardy type inequalities. The version of horizontal weighted Hardy-Rellich type inequalities was obtained on the stratified Lie groups as a consequence of this inequality Sobolev type spaces are defined on stratified Lie groups and the embedding theorems are proved for these functional spaces. Also, we have obtained the subelliptic Picone type identities, as a result, we proved the Hardy and Rellich type inequalities for the anisotropic p-sub-Laplacians. Moreover, analogues of Hardy type inequalities with multiple singularities and many-particle Hardy type inequalities are obtained on the stratified groups.

- *Hardy and Rellich type inequalities and the sub-Laplacian fundamental solutions.* In the third direction, we investigate the following type of Hardy inequalities. Generalised weighted L^p-Hardy, L^p-Rellich, and L^p-Caffarelli-Kohn-Nirenberg type inequalities with boundary terms are obtained on the stratified groups. As consequences, most of the Hardy type inequalities and the Heisenberg-Pauli-Weyl type uncertainty principles are recovered on the stratified groups. Moreover, a weighted L^2-Rellich type inequality with the boundary term is obtained. We also present Hardy and Rellich inequalities for the sub-Laplacians in terms of their fundamental solutions on the quaternion Heisenberg group.

- *Weighted Hardy and Rellich type inequalities for general vector fields.* In this direction, we study the weighted Hardy and Rellich inequalities for general vector fields without a group structure. We establish the weighted anisotropic Hardy and Rellich type inequalities with boundary terms for general (real-valued) vector fields. As consequences, we derive new as well as many of the fundamental Hardy and Rellich type inequalities which are known in different settings.

Theoretical and practical significance of the results. This study has mainly fundamental character, makes a valuable contribution to the development of theory of functional analysis on nilpotent Lie groups and theory of differential equations, and promotes raising the image of the Republic of Kazakhstan in the

scientific world. Obtained results will be applied to solving various problems in theoretical physics.

Publications. On the topic of the thesis 15 papers were published, including 3 publications in scientific journals included in the list recommended by the Committee on the Control of Education and Science of the MES RK for publication of the main scientific results of scientific activities, 4 publications in a high-ranking scientific journal, indexed in the Thomson database Reuters, 8 publications in materials of international conferences, including 1 publication in materials of a foreign international conference**.**

The results on the topic of the thesis were published in the following papers:

Publication in the high-ranking scientific journals

1 Sabitbek B., Suragan D. Horizontal Weighted Hardy–Rellich Type Inequalities on Stratified Lie Groups // Complex Analysis and Operator Theory. – 2018, - V. 12, - P. 1469-1480. (Scopus, Web of Science, Q2).

2 Sabitbek B., Suragan D. On green functions for Dirichelet sub-Laplacians on a Quaternion Heisenberg group // Mathematical Modelling of Natural Phenomena. – 2018 –V. 13, - No. 4. (Scopus, Web of Science, Q3).

3 Ruzhansky M., Sabitbek B., Suragan D. Weighted Lp-Hardy and Lp-Rellich inequalities with boundary terms on stratified Lie groups // Revista Matematica Complutense. – 2019, – Vol. 32, - Issue 1, - P. 19–35. (Scopus, Web of Science, Q1).

4 Ruzhansky M., Sabitbek B., Suragan D. Weighted anisotropic Hardy and Rellich type inequalities for general vector fields // Nonlinear Differential Equations and Applications (NoDEA). – 2019, - V. 26, - No. 13. (Scopus, Web of Science, Q1).

CCES

1 Sabitbek B. Embedding theorem of Sobolev type spaces on stratified Lie groups // Mathematical Journal. – 2016, - V. 16, - No. 3(61), - P. 166-180.

2 Sabitbek B., Suragan D. Hardy and Rellich type inequalities on the complex affine group // Eurasian Mathematical Journal. – 2017. - V. 8, - No. 2. - P. 31-39.

3 Kalmenov T.Sh., Sabitbek B. On Hardy and Rellich type inequalities for the Grushin operator // Mathematical Journal. – 2018, - V. 18, - No. 2(68), - P. 133-142.

Publications in materials of international conferences

1 Sabitbek B. Hardy and Rellich type inequalities on Complex Affine group // Конференция "Весомая оценка дифференциальных и интегральных

операторов и их приложение", Евразиский национальный университет им. Л.Н.Гумилев, – 2017, С. 50-51.

2 Sabitbek B. On Hardy and Rellich type inequalities for Grushin operator // Тезисы докладов: ежегодная научная апрельская конференции института математики и математического моделирования, посвящённая дню науки и научный семинар “Дифференциальные операторы и моделирование сложных систем” посвящённый 70-летнему юбилею профессора Мувашархана Танабаевича Дженалиева. – 2017,- С. 166-167.

3 Sabitbek B. An embedding theorem of Sobolev type spaces on stratified Lie groups // International Conference on Analysis and Applied Mathematics (ICAAM 2016)", Almaty, 2016.

4 Sabitbek B. Horizontal weighted Hardy-Rellich type inequalities // Book of Abstracts, ISAAC Congress 2017, Векше, Швеция.

5 Sabitbek B. On Hardy and Rellich type inequalities for an Engel-type operator // AIP Conference Proceedings, - 2017.

6 Sabitbek B., Suragan D., Yesserkegenov N. Improved critical Hardy inequalities on 2-dimensional quasi-balls // AIP Conference Proceedings, - 2017.

7 Kalmenov T.Sh., Sabitbek B., Suragan D. An inequality for Schatten p-norms of compact operators // AIP Conference Proceedings, - 2016.

8 Kalmenov T.Sh., Sabitbek B. A Boundary Condition of the Volume Potential for Strongly Elliptic Differential Equations // Springer Proceedings in Mathematics and Statistics, - 2017.

The structure and scope of the thesis. The PhD thesis includes a title page, content, introduction, four chapters, conclusion and list of references, consisting of 83 titles. The total volume of the thesis is 105 pages.

The main content of the thesis. The introduction reflects the substantiation of the relevance of the topic of the dissertation, the purpose, object, subject, research objectives, the substantiation of the scientific novelty of the work, its theoretical and practical significance, the scientific principles to be defended, the number of publications available.

In chapter 1, we study the geometric Hardy and Hardy-Sobolev inequalities on the half-spaces by the formula

$$\int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi \geq \frac{1}{4}\int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^2}{dist(\xi, \partial\mathbb{H}^+)^2}|u|^2 d\xi,$$

and

$$\left(\int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi - \frac{1}{4}\int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^2}{dist(\xi, \partial\mathbb{H}^+)^2}|u|^2 d\xi\right)^{1/2} \geq C\left(\int_{\mathbb{H}^+} |u|^2 d\xi\right)^{\frac{Q-2}{2Q}},$$

where $dist(\xi, \partial\mathbb{H}^+)$ is the Euclidean distance to the boundary and the angle function

$$\mathcal{W}(\xi) := \left(\sum_{i=1}^{n} \langle X_i(\xi), \nu\rangle^2 + \langle Y_i(\xi), \nu\rangle^2\right)^{\frac{1}{2}}.$$

In chapter 2, we study the following horizontal version of Hardy type inequalities

$$\int_{\mathbb{G}} |\nabla_H u(x)|^2 dx \geq \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{G}} \frac{|u(x)|^2}{|x'|^2} dx,$$

where $|\cdot|$ is the Euclidean norm and ∇_H is a horizontal gradient.

In chapter 3, we investigate the following type of Hardy inequalities

$$\int_{\mathbb{G}} |\nabla_H u(x)|^2 \geq \left(\frac{Q-2}{2}\right)^2 \int_{\mathbb{G}} \frac{|u(x)|^2}{(d(x))^2} dx,$$

where Q is the homogeneous dimension of the stratified group $\mathbb{G}$ and $d(x)$ is the so-called $\mathcal{L}$-gauge, which is a particular homogeneous quasi-norm obtained from the fundamental solution of the sub-Laplacian.

In chapter 4, we study the weighted Hardy and Rellich inequalities for general vector fields without a group structure as

$$\int_\Omega W(x)|\nabla_X u|^p dx \geq \int_\Omega H(x)|u|^p dx, \ \forall u \in C_0^1(\Omega),$$

and

$$\int_\Omega W(x)|\mathcal{L}_X u|^p dx \geq \int_\Omega H(x)|u|^p dx, \ \forall u \in C_0^\infty(\Omega).$$